Bistable kinetic model driven by correlated noises: Steady-state analysis

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A simple rule to obtain the Fokker-Planck equation for a general one-dimensional system driven by correlated Gaussian white noises is proved by the functional method. The Fokker-Planck equation obtained in this paper is applied to the bistable kinetic model. We find the following for the steady state. (1) In the α-D parameter plane (α is the strength of the additive noise and D is the multiplicative noise strength), the critical curve separating the unimodal and bimodal regions of the stationary probability distribution (SPD) of the model is shown to be affected by λ, the degree of correlation of the noises. As λ is increased, the area of the bimodal region in the α-D plane is contracted. (2) When we take a point fixed in the α-D plane and increase λ, the form of the SPD changes from a bimodal to a unimodal structure. (3) The positions of the extreme value of the SPD of the model sensitively depend on the strength of the multiplicative noise, and weakly depend on the additive noise strength. (4) For $\lambda=1$, the case of perfectly correlated noises, when the parameters α and D take values in the neighborhood of the line $\alpha=D$ in the α-D plane, the SPD's corresponding to the points $\alpha/D>1$ and $\alpha/D<1$ exhibit a very different shape of divergence. Therefore, the ratio $\alpha/D=1$ plays the role of a "critical ratio."

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I. INTRODUCTION

On the level of a Langevin-type description of a dynamical system, the presence of correlation between noises changes the dynamics of the system [1]. In Ref. [1], a bistable kinetic model under the simultaneous influence of additive and multiplicative Gaussian white noises is considered. The authors of Ref. [1] pointed out correctly that the transition between unimodal and bimodal stationary distribution is strongly influenced by the correlations between both noises. However, the statistical properties for the systems driven by correlated additive and multiplicative noises have still not been investigated because the method given in Ref. [1] cannot provide a correct foundation with which the effects of correlation of the noises will be studied quantitatively [2].

It must be pointed out that Fox discussed non-Markovian, Gaussian, N-component stochastic processes with correlation between the noise components by a correlation time expansion [3]. He obtained an explicit result to first order in the correlation time and systematically provide any higher order correction. Singh showed that the correlation between the quantum noise for a homogeneously broadened two-mode ring laser at line center gives a nonzero contribution of the order n_0^{-1} [4]; here n_0 denotes the mean number of photons in the laser cavity at threshold. In our opinion the effects of the correlation of quantum noises between the laser modes may be of interest to the problem of laser physics. In the recent years, Fedchenia described the influence of two additive correlated noise effects on a two-dimensional quadratic-nonlinear system describing the behavior of two hydrodynamic modes. Using the method of local expansions, Fedchenia obtained an approximate stationary distribution function [5,6]. More recently, Zhu investigated theoretically the statistical fluctuations of a single-mode laser with correlations between additive and multiplicative white-noise terms. The mean, variance, and skewness of the steady-state laser intensity are calculated by Zhu through a one-dimensional laser equation [7].

In this paper we propose a simple method to obtain the Fokker-Planck equation (FPE) corresponding to the Langevin equation driven by correlated noises (it may be multiplicative noise) with an arbitrary degree of correlation in Sec. II. Then, in Sec. III, the FPE obtained above is used in a typical one-dimensional dynamical system, the bistable kinetic system, driven by correlated additive and multiplicative Gaussian white noises. The stationary probability distribution (SPD) of the state variable for the system is derived. In Sec. IV several interesting conclusions about the transition between the unimodal and bimodal structures for the SPD are drawn.

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II. FOKKER-PLANCK EQUATION CORRESPONDING TO THE LANGEVIN EQUATION DRIVEN BY CORRELATED GAUSSIAN WHITE NOISES

Consider a one-dimensional Langevin equation (LE) with two correlated Gaussian white noises $\epsilon(t)$ and $\Gamma(t)$:

$$\dot{x}=h(x)+g_1(x)\epsilon(t)+g_2(x)\Gamma(t) \ . \tag{1}$$

In the paper we assume Eq. (1) to be the Stratonovich stochastic differential equation. In Eq. (1), $\epsilon(t)$ and $\Gamma(t)$ are Gaussian white noises with zero mean and

$$\langle \epsilon(t)\epsilon(t')\rangle = 2D\delta(t-t') \ ,$$
$$\langle \Gamma(t)\Gamma(t')\rangle = 2\alpha\delta(t-t') \ , \tag{2a}$$
$$\langle \epsilon(t)\Gamma(t')\rangle = \langle \Gamma(t)\epsilon(t')\rangle = 2\lambda\sqrt{D\alpha}\delta(t-t') \ . \tag{2b}$$

λ denotes the degree of correlation between the noises $\epsilon(t)$ and $\Gamma(t)$. D and α are the strength of the noises $\epsilon(t)$ and $\Gamma(t)$, respectively.

To obtain the FPE from the LE (1) with (2), we have proved the following simple rule (for the details for the proof of this rule, see the Appendix): Equation (1) with (2) can be transformed into a stochastic equivalent Stratonovich stochastic differential equation (i.e., leads to the same FPE)

$$\dot{x}=h(x)+G(x)\tilde{\Gamma}(t) \ , \tag{3}$$

in which $\tilde{\Gamma}(t)$ is Gaussian white noise with zero mean and

$$\langle \tilde{\Gamma}(t)\tilde{\Gamma}(t')\rangle = 2\delta(t-t') \tag{4}$$

and $G(x)$ is determined by the following simple procedure: Let the correlation of $G(x)\tilde{\Gamma}(t)$ in Eq. (3) be equal to the correlation of

$$g_1(x)\epsilon(t)+g_2(x)\Gamma(t)$$

in Eq. (1):

$$\langle G(x)\tilde{\Gamma}(t)G(x)\tilde{\Gamma}(t')\rangle = \langle [g_1(x)\epsilon(t)+g_2(x)\Gamma(t)] \times [g_1(x)\epsilon(t')+g_2(x)\Gamma(t')]\rangle \ .$$

This gives the relation

$$[G(x)]^2\langle \tilde{\Gamma}(t)\tilde{\Gamma}(t')\rangle = [g_1(x)]^2\langle \epsilon(t)\epsilon(t')\rangle + g_1(x)g_2(x)\langle \epsilon(t)\Gamma(t')\rangle + g_2(x)g_1(x)\langle \Gamma(t)\epsilon(t')\rangle + [g_2(x)]^2\langle \Gamma(t)\Gamma(t')\rangle \ .$$

Using (2) and (4), the above relation leads to the required expression

$$G(x)=\{D[g_1(x)]^2+2\lambda\sqrt{D\alpha}g_1(x)g_2(x) + \alpha[g_2(x)]^2\}^{1/2} \ . \tag{5}$$

The FPE corresponding to (3) with (4) is given by [8,9]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}A(x)P(x,t)+\frac{\partial^2}{\partial x^2}B(x)P(x,t) \ , \tag{6}$$

where

$$A(x)=h(x)+G'(x)G(x) \tag{7a}$$

and

$$B(x)=[G(x)]^2 \ . \tag{7b}$$

The prime in Eq. (7a) and below denotes the derivation with respect to x.

Using expression (5), we rewrite Eqs. (7a) and (7b) as

$$A(x)=h(x)+Dg_1(x)g_1'(x)+\lambda\sqrt{D\alpha}g_1(x)g_2'(x) + \lambda\sqrt{D\alpha}g_1'(x)g_2(x)+\alpha g_2(x)g_2'(x) \tag{8}$$

and

$$B(x)=D[g_1(x)]^2+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha[g_2(x)]^2 \ . \tag{9}$$

The SPD of FPE (6) is given by [8,9]

$$P_{st}(x)=\frac{N}{B(x)}\exp\left\{\int^x \frac{A(x')}{B(x')}dx'\right\} = \frac{N}{\{D[g_1(x)]^2+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha[g_2(x)]^2\}^{1/2}}\exp\left\{\int^x \frac{h(x')dx'}{D[g_1(x')]^2+2\lambda\sqrt{D\alpha}g_1(x')g_2(x')+\alpha[g_2(x')]^2}\right\} \ . \tag{10}$$

In addition, the extrema of $P_{st}(x)$ obey a general equation $A(x)-B'(x)=0$ or $h(x)-G'(x)G(x)=0$. Using expression (5), this leads to

$$h(x)-\{Dg_1(x)g_1'(x)+\lambda\sqrt{D\alpha}g_1(x)g_2'(x) + \lambda\sqrt{D\alpha}g_1'(x)g_2(x)+\alpha g_2(x)g_2'(x)\}=0 \ . \tag{11}$$

III. THE BISTABLE KINETIC SYSTEM DRIVEN BY CORRELATED ADDITIVE AND MULTIPLICATIVE NOISES

Now we apply the theory developed above to an important one-dimensional system, the bistable kinetic system driven by additive and multiplicative noises, and assume the dimensionless form

$$\dot{x}=x-x^3+x\epsilon(t)+\Gamma(t) \ . \tag{12}$$

The noises $\epsilon(t)$ and $\Gamma(t)$ are the same as in Eq. (1).

In order to obtain the SPD for Eq. (12), we compare Eq. (12) with Eq. (1) and get

$$h(x)=x-x^3 \ , \quad g_1(x)=x \ , \quad g_2(x)=1 \ . \tag{13}$$

Substituting (13) into (10) and making the integral operation, we obtain the SPD for Eq. (12) [10]:

$$P_{st}(x)=\begin{cases} N[Dx^2+2\lambda\sqrt{D\alpha}x+\alpha]^{C-1/2}\exp\left\{f(x)+\frac{E}{[(1-\lambda^2)D\alpha]^{1/2}}tg^{-1}\frac{Dx+\lambda\sqrt{D\alpha}}{[(1-\lambda^2)D\alpha]^{1/2}}\right\} & \text{for } 0\le\lambda<1 \quad (14)\\ \tilde{N}[Dx^2+2\sqrt{D\alpha}x+\alpha]^{\tilde{C}-1/2}\exp\left\{\tilde{f}(x)-\frac{\tilde{E}}{Dx+\sqrt{D\alpha}}\right\} & \text{for } \lambda=1\ . \quad (15)\end{cases}$$

Here,

$$f(x)=2\lambda\left[\frac{\alpha}{D}\right]^{1/2}\frac{x}{D}-\frac{x^2}{2D}\ ,\quad E=\lambda\left[\frac{\alpha}{D}\right]^{1/2}\left[(4\lambda^2-1)\frac{\alpha}{D}-1\right]\ ,\quad C=\frac{1}{2D}\left[1-(4\lambda^2-1)\frac{\alpha}{D}\right] \tag{16}$$

and

$$\tilde{f}(x)=2\left[\frac{\alpha}{D}\right]^{1/2}\frac{x}{D}-\frac{x^2}{2D}\ ,\quad \tilde{E}=\left[\frac{\alpha}{D}\right]^{1/2}\left[\frac{\alpha}{D}-1\right]\ ,\quad \tilde{C}=\frac{1}{2D}\left[1-\frac{3\alpha}{D}\right]\ . \tag{17}$$

The extrema of the SPD are determined by the following equation of third order:

$$x^3+(D-1)x+\lambda\sqrt{D\alpha}=0 \quad \text{for } 0\le\lambda\le 1\ . \tag{18}$$

The critical curve separating the unimodal and bimodal regions in the α-D plane is described by

$$\tfrac{1}{4}\lambda^2D\alpha+\tfrac{1}{27}(D-1)^3=0\ . \tag{19}$$

IV. STEADY-STATE ANALYSIS: CONCLUSIONS

A. The case of $0\le\lambda<1$

By virtue of the results obtained above for the bistable kinetic model, the expression of the SPD (14) for $0\le\lambda<1$ and the equation of the critical curve in the α-D plane (19), we have plotted the critical curves in Fig. 1 and the curves of the SPD in Figs. 2–5. The conclusions that can be drawn from these figures are as follows.

(1) For the bistable kinetic model the presence of correlation between the noises makes the critical curve separating the unimodal and bimodal regions in the α-D parameter plane affected not only by the multiplicative noise but also by the additive noise, as can be seen from Fig. 1. The Figure also shows that in the case of uncorrelated noises ($\lambda=0$), the critical curve becomes a horizontal line, that is, independent of the strength of the additive noise.

(2) As the degree of the correlation of the noises λ is increased, the area of the bimodal region in the α-D plane is contracted, as can be seen from Fig. 1.

(3) It is interesting to point out that when we increase λ, the SPD of the system corresponding to a fixed point in the α-D plane (the point I in Fig. 1) experiences the transition from a bimodal to a unimodal structure, as shown in Fig. 2. Similarly, Fig. 3 corresponds to the fixed point J in Fig. 1.

(4) From Fig. 2 we see that if the noises are uncorrelated, $\lambda=0$, the SPD of the system exhibits a symmetry bimodal structure as usual. But when the noises are corre-

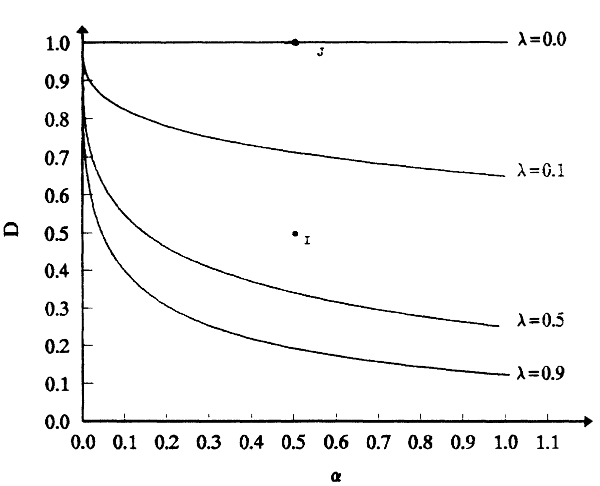

FIG. 1. The critical curves in the α-D parameter plane. The SPD corresponding to points I and J for different λ are shown in Figs. 2 and 3, respectively.

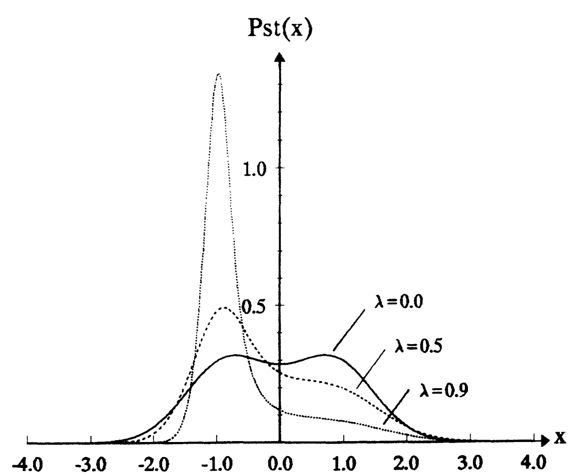

FIG. 2. The SPD of the bistable kinetic model for $\lambda<1$ [Eq. (14)]. $\alpha=D=0.5$ is fixed (relative to point I in Fig. 1) and $\lambda=0.0$, 0.5, and 0.9, respectively.

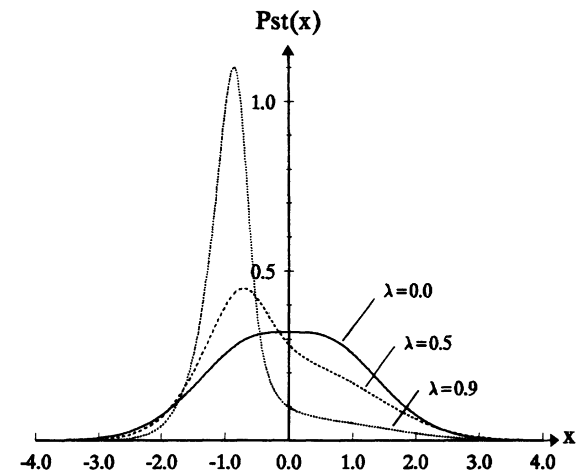

FIG. 3. The SPD of the bistable kinetic model for $\lambda<1$ [Eq. (14)]. $\alpha=0.5$ and $D=1$ are fixed (corresponding to point J in Fig. 1) and $\lambda=0.0$, 0.5, and 0.9, respectively.

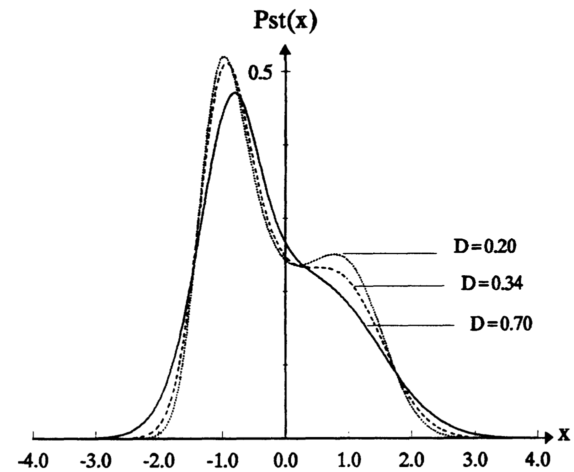

FIG. 5. The SPD of the bistable kinetic model for $\lambda<1$ [Eq. (14)]. $\alpha=\lambda=0.5$ and $D=0.70$, 0.34, and 0.20, respectively.

lated, this symmetry is destroyed. The larger the λ, the stronger the destruction of the symmetry of the SPD.

(5) Figure 3 shows the same thing as Fig. 2, but the SPD for $\lambda=0$ shows that the critical state has set in due to the strength of multiplicative noise $D=D_c=1$, i.e., it takes the value of critical noise intensity.

(6) When the degree of the correlation of the noises and the strength of the multiplicative noise are fixed (λ and D fixed), as the additive noise intensity α is increased, the SPD of the system experiences the transition from a bimodal to a unimodal structure, as shown in Fig. 4. From the Figure we see that the position of the extrema of the SPD is weakly affected by the strength of the additive noise α; however, its high may be affected intensively by α.

(7) It is contrary to Fig. 4 that when λ and α are fixed, and changing the multiplicative noise intensity D, the SPD of the system changes its position of the extrema with D intensively, but the high of the extremum of the SPD is weakly affected by D, as can be seen from Fig. 5.

B. The case of $\lambda=1$

By virtue of the expression of the SPD (15) for $\lambda=1$, we have plotted the curves of the SPD in Figs. 6–9. The conclusions that can be drawn from these figures are as follows.

(1) Equations (15) and (17) show that when the strength of the additive noise is equal to the multiplicative noise strength (if $\alpha=D$, then $\tilde{E}=0$), the SPD exhibits divergence at $x=-(\alpha/D)^{1/2}=-1$ due to $\tilde{C}<0$, as shown by Fig. 6. In this figure we let $\alpha=D=0.5$.

(2) The line $\alpha=D$ in the α-D plane separates the plane into two parts. The region with $\alpha>D$ makes $\tilde{E}>0$, while $\alpha<D$ makes $\tilde{E}<0$. We see from (17) that in the neighborhood of the line $\alpha=D$ the parameter $\tilde{C}$ is still nega-

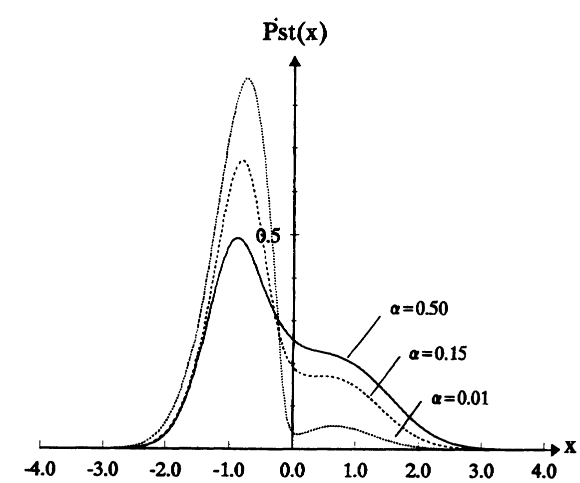

FIG. 4. The SPD of the bistable kinetic model for $\lambda<1$ [Eq. (14)]. $D=\lambda=0.5$ and $\alpha=0.50$, 0.15, and 0.01, respectively.

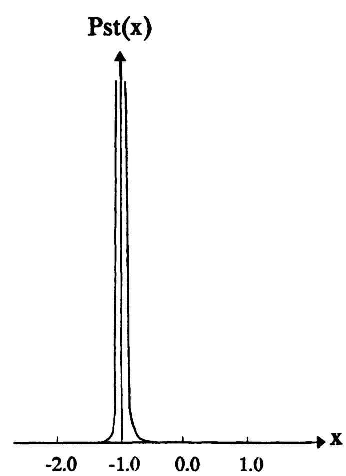

FIG. 6. The SPD of the bistable kinetic model for $\lambda=1$ [Eq. (15)]. $\alpha=D=0.5$. [$P_{st}(x)$ denotes the relative probability density.]

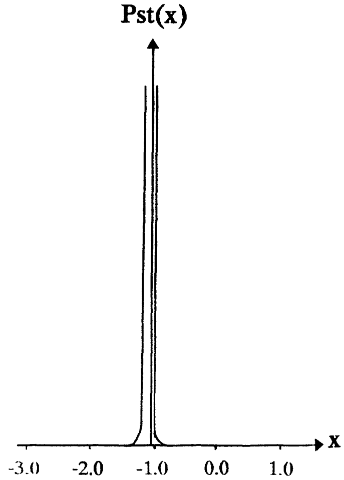

FIG. 7. The SPD of the bistable kinetic model for $\lambda=1$ [Eq. (15)]. $\alpha=0.55$ and $D=0.50$. [$P_{st}(x)$ denotes the relative probability density.]

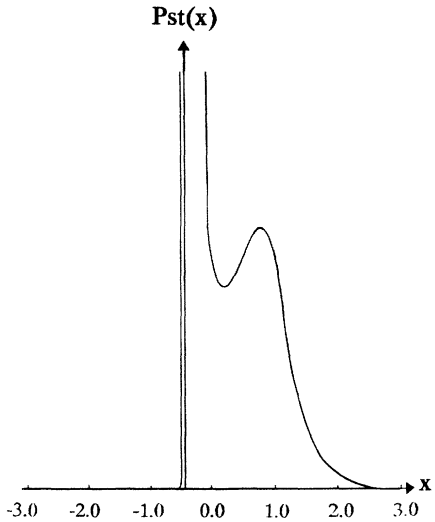

FIG. 9. The SPD of the bistable kinetic model for $\lambda=1$ [Eq. (15)]. $\alpha=0.05$ and $D=0.25$. [$P_{st}(x)$ denotes the relative probability density.]

tive. The above facts lead the SPD to diverge at a point $x=-(\alpha/D)^{1/2}$. For example, when $\alpha=0.55$ and $D=0.50$, so that $\tilde{E}>0$ and $\tilde{C}<0$, the combination of the factor $\exp(-\tilde{E}/(Dx+\sqrt{D\alpha})$ with the factor

$$(Dx^2+2\sqrt{D\alpha}x+\alpha)^{\tilde{C}-1/2}$$

leads to the extraordinary asymmetry of the SPD around $x=-(\alpha/D)^{1/2}$, as can be seen from Fig. 7. On the contrary, as $\alpha=0.45$ and $D=0.50$, so that $\tilde{E}<0$ and $\tilde{C}<0$, the asymmetry of the SPD around $x=-(\alpha/D)^{1/2}$ changes, as shown by Fig. 8.

(3) It follows from (18) and (19) that the critical curve for $\lambda=1$ makes no qualitative change. In the parameter plane (α,D) there still exist two regions for $\lambda=1$, that is, the bimodal and unimodal regions. Figure 9 shows a bimodal structure of the SPD.

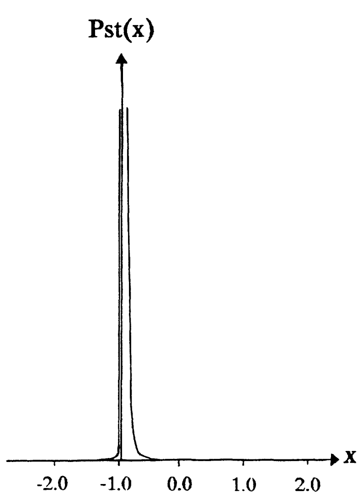

FIG. 8. The SPD of the bistable kinetic model for $\lambda=1$ [Eq. (15)]. $\alpha=0.45$ and $D=0.50$. [$P_{st}(x)$ denotes the relative probability density.]

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APPENDIX: THE PROOF OF THE STOCHASTIC EQUIVALENCE OF EQS. (1) AND (2) AND EQS. (3)–(5)

To do this, we prove that Eqs. (1) and (2) and Eqs. (3)–(5) both lead to the same Fokker-Planck equation. The stochastic Liouville equation corresponding to Eq. (1) reads

$$\frac{\partial\rho(x,t)}{\partial t}=-\frac{\partial}{\partial x}[h(x)+g_1(x)\epsilon(t)+g_2(x)\Gamma(t)]\rho(x,t) \ . \tag{A1}$$

Here we have considered an ensemble of systems in x space obeying Eq. (1) for a given realization of $\epsilon(t)$ and $\Gamma(t)$ and different initial conditions. This ensemble is represented by a density $\rho(x,t)$ which evolves in time according to Eq. (A1). It is well known that $\rho(x,t)$ is just the average of $\delta(x(t)-x)$ over the initial conditions, where $x(t)$ is a solution of Eq. (1) for a given realization of $\epsilon(t)$ and $\Gamma(t)$ and for a given initial condition, while x is a point in state space. Because the probability density $P(x,t)$ is obtained by averaging $\rho(x,t)$ over the realizations of $\epsilon(t)$ and $\Gamma(t)$ (this is known as van Kampen's lemma [11]), we have

$$P(x,t)=\langle\delta(x(t)-x)\rangle \ . \tag{A2}$$

Here the average $\langle\ \rangle$ is taken over initial conditions and over the realizations of $\epsilon(t)$ and $\Gamma(t)$. Because Eq. (A1) expresses the variation of $\rho(x,t)$ with time at a fixed point x; therefore, $h(x)$, $g_1(x)$, and $g_2(x)$ are given functions independent of $\epsilon(t)$ and $\Gamma(t)$, while $\rho(x,t)$ is a functional of $\epsilon(t)$ and $\Gamma(t)$ defined through Eq. (A1). So that the evolution equation for the probability density $P(x,t)$ is obtained by averaging Eq. (A1):

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}h(x)P(x,t)-\frac{\partial}{\partial x}g_1(x)\langle \epsilon(t)\delta(x(t)-x)\rangle-\frac{\partial}{\partial x}g_2(x)\langle \Gamma(t)\delta(x(t)-x)\rangle\ . \qquad \text{(A3)}$$

The average which remains in Eq. (A3) may be calculated for Gaussian noises $\epsilon(t)$ and $\Gamma(t)$ by a functional formula, the Novikov theorem [12]:

$$\langle \zeta_k\phi[\zeta_1,\zeta_2]\rangle=\int_0^t dt'\gamma_{kl}(t,t')\left\langle \frac{\delta(\delta(x(t)-x))}{\delta\zeta_l(t')}\right\rangle \quad (k,l=1,2)\ , \qquad \text{(A4)}$$

where $\phi[\zeta_1,\zeta_2]$ is a functional of ζ_1 and ζ_2 and $\gamma_{kl}=\langle \zeta_k(t)\zeta_l(t')\rangle$ are its correlation functions. Now we use the above theorem to the calculation of the averages $\langle \epsilon(t)\delta(x(t)-x)\rangle$ and $\langle \Gamma(t)\delta(x(t)-x)\rangle$ in Eq. (A3). In our situation, ζ_1 and ζ_2 are the correlated noises $\epsilon(t)$ and $\Gamma(t)$ with correlation functions, Eq. (2). According to Eq. (A4), we have

$$\langle \epsilon(t)\delta(x(t)-x)\rangle=\int_0^t dt'\gamma_{11}(t,t')\left\langle \frac{\delta(\delta(x(t)-x))}{\delta\epsilon(t')}\right\rangle+\int_0^t dt'\gamma_{12}(t,t')\left\langle \frac{\delta(x(t)-x))}{\delta\Gamma(t')}\right\rangle\ . \qquad \text{(A5)}$$

This leads to

$$\langle \epsilon(t)\delta(x(t)-x)\rangle=-\frac{\partial}{\partial x}\int_0^t dt'\gamma_{11}(t,t')\left\langle \delta(x(t)-x)\frac{\delta x(t)}{\delta\epsilon(t')}\right\rangle-\frac{\partial}{\partial x}\int_0^t dt'\gamma_{12}(t,t')\left\langle \delta(x(t)-x)\frac{\delta x(t)}{\delta\Gamma(t')}\right\rangle\ . \qquad \text{(A6)}$$

Using Eqs. (2a) and (2b), that is,

$$\gamma_{11}=\langle \epsilon(t)\epsilon(t')\rangle=2D\delta(t-t')\ ,$$
$$\gamma_{12}=\langle \epsilon(t)\Gamma(t')\rangle=2\lambda\sqrt{D\alpha}\delta(t-t')\ ,$$

we obtain [13] from Eq. (A6),

$$\langle \epsilon(t)\delta(x(t)-x)\rangle=-\frac{\partial}{\partial x}Dg_1(x)P(x,t)-\frac{\partial}{\partial x}\lambda\sqrt{D\alpha}g_2(x)P(x,t)\ . \qquad \text{(A7)}$$

Similarly, also using (2a) and (2b),

$$\gamma_{22}=\langle \Gamma(t)\Gamma(t')\rangle=2\alpha\delta(t-t')\ ,$$
$$\gamma_{21}=\langle \Gamma(t)\epsilon(t')\rangle=2\lambda\sqrt{D\alpha}\delta(t-t')\ , \qquad \text{(A8)}$$

we get

$$\langle \Gamma(t)\delta(x(t)-x)\rangle=-\frac{\partial}{\partial x}\lambda\sqrt{D\alpha}g_1(x)P(x,t)-\frac{2}{\partial x}\alpha g_2(x)P(x,t)\ . \qquad \text{(A9)}$$

Substituting (A7) and (A9) into (A3), we finally obtain a Fokker-Planck equation corresponding to Eqs. (1) and (2):

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}\{h(x)+Dg_1(x)g_1'(x)+\lambda\sqrt{D\alpha}[g_1(x)g_2'(x)+g_1'(x)g_2(x)]+\alpha g_2(x)g_2'(x)\}P(x,t)+\frac{\partial^2}{\partial x^2}[Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)]P(x,t)\ . \qquad \text{(A10)}$$

Following the same method, we get from (3)–(5) the Fokker-Planck equation

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t}&=-\frac{\partial}{\partial x}h(x)P(x,t)-\frac{\partial}{\partial x}G(x)\langle \tilde{\Gamma}(t)\delta(x(t)-x)\rangle\\&=-\frac{\partial}{\partial x}h(x)P(x,t)+\frac{\partial}{\partial x}G(x)\frac{\partial}{\partial x}\int_0^t dt'\gamma(t,t')\left\langle \delta(x(t)-x)\frac{\delta x(t)}{\delta\tilde{\Gamma}(t')}\right\rangle\\&=-\frac{\partial}{\partial x}h(x)P(x,t)+\frac{\partial}{\partial x}G(x)\frac{\partial}{\partial x}G(x)P(x,t)\\&=-\frac{\partial}{\partial x}h(x)P(x,t)-\frac{\partial}{\partial x}\{Dg_1(x)g_1'(x)+\lambda\sqrt{D\alpha}[g_1(x)g_2'(x)+g_1'(x)g_2(x)]+\alpha g_2(x)g_2'(x)\}P(x,t)\\&\quad+\frac{\partial^2}{\partial x^2}[Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)]P(x,t)\ .\end{aligned}$$

This is the same Fokker-Planck equation as Eq. (A10). It is evident from the above proof of the rule (3)–(5) that the greatest advantage of this rule lies in its simplicity and it can be used for the dynamical system driven by an arbitrary number of correlated noises.

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